$$\widehat{\phi}'(k) = -\left(k\frac{\partial \widehat{G}(0)}{\partial k} + \frac{k^2}{2}\frac{\partial^2 \widehat{G}(0)}{\partial k^2} + ... \right) \widehat{\phi}(k) \quad . \tag{6.26}$$

Simple algebraic manipulations, which consist in substituting these relations into those of Sect. 3.3.1, lead to the same results as those that have just been explained for the physical space. To do this, we need simply identify groups of the form $k^n \widehat{\phi}(k)$ with the term $\partial^n \phi(x)/\partial x^n$.

Iterative Deconvolution. If the filter kernel G has an inverse G^{-1}, the latter can also be obtained using the following expansion [319, 320]:

$$G^{-1} \star \phi = \sum_{p=0,\infty} (I - G)^p \star \phi \quad , \tag{6.27}$$

yielding the following reconstruction for the defiltered variable ϕ:

$$\phi = \overline{\phi} + (\overline{\phi} - \overline{\overline{\phi}}) + (\overline{\phi} - 2\overline{\overline{\phi}} + \overline{\overline{\overline{\phi}}}) + ... \quad , \tag{6.28}$$

or equivalently

$$\phi' = (\overline{\phi} - \overline{\overline{\phi}}) + (\overline{\phi} - 2\overline{\overline{\phi}} + \overline{\overline{\overline{\phi}}}) + ... \tag{6.29}$$

The series are known to be convergent if $|I - G| < 1$. A practical model is obtained by truncating the expansion at a given power. Stolz and Adams [319] recommend using a fifth-order expansion.

6.1.2 Non-linear Models

There are a number of ways of deriving non-linear models: Horiuti [140], Speziale [315], Yoshizawa [359], and Wong [348] start with an expansion in a small parameter, while Lund and Novikov [205] use the mathematical properties of the tensors considered. It is this last approach that will be described first, because it is the one that best reveals the difference with the functional models. Kosovic's simplified model [167] and Wong's dynamic model [348] are then described.

Generic Model of Lund and Novikov. We assume that the deviator of the subgrid tensor can be expressed as a function of the resolved velocity field gradients (and not the velocity field itself, to ensure the Galilean invariance property), the unit tensor, and the square of the cutoff length $\overline{\Delta}$:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} \equiv \tau^{\mathrm{d}}_{ij} = \mathcal{F}(\overline{S}_{ij}, \overline{\Omega}_{ij}, \delta_{ij}, \overline{\Delta}^2) \quad . \tag{6.30}$$

The isotropic part of τ is not taken into account, and is integrated in the pressure term because $\overline{S}$ and $\overline{\Omega}$ have zero traces. To simplify the expansions in the following, we use the reduced notation:

$$\overline{S}\,\overline{\Omega} = \overline{S}_{ik}\overline{\Omega}_{kj}, \quad \mathrm{tr}(\overline{S}\,\overline{\Omega}^2) = \overline{S}_{ij}\overline{\Omega}_{jk}\overline{\Omega}_{ki}$$

The most general form for relation (6.30) is a polynomial of infinite degree of tensors whose terms are of the form $\overline{S}^{a_1}\overline{\Omega}^{a_2}\overline{S}^{a_3}\overline{\Omega}^{a_4}...$, where the a_i are positive integers. Each terms in the series is multiplied by a coefficient, which is itself a function of the invariants of $\overline{S}$ and $\overline{\Omega}$. This series can be reduced to a finite number of linearly independent terms by the Cayley- Hamilton theorem. Since the tensor τ^{d} is symmetrical, we retain only the symmetrical terms here. The computations lead to the definition of eleven tensors, $m_1, ..., m_{11}$, with which $I_1, ..., I_6$ are associated:

$$\begin{aligned}
m_1 &= \overline{S}, & m_2 &= \overline{S}^2, \\
m_3 &= \overline{\Omega}^2, & m_4 &= \overline{S}\,\overline{\Omega} - \overline{\Omega}\,\overline{S}, \\
m_5 &= \overline{S}^2\overline{\Omega} - \overline{\Omega}\overline{S}^2, & m_6 &= Id, \\
m_7 &= \overline{S}\,\overline{\Omega}^2 + \overline{\Omega}^2\overline{S}, & m_8 &= \overline{\Omega}\,\overline{S}\,\overline{\Omega}^2 - \overline{\Omega}^2\overline{S}\,\overline{\Omega}, \\
m_9 &= \overline{S}\,\overline{\Omega}\,\overline{S}^2 - \overline{S}^2\overline{\Omega}\,\overline{S}, & m_{10} &= \overline{S}^2\overline{\Omega}^2 + \overline{\Omega}^2\overline{S}^2, \\
m_{11} &= \overline{\Omega}\,\overline{S}^2\overline{\Omega}^2 - \overline{\Omega}^2\overline{S}^2\overline{\Omega}, & &
\end{aligned} \tag{6.31}$$

$$\begin{aligned}
I_1 &= \mathrm{tr}(\overline{S}^2), & I_2 &= \mathrm{tr}(\overline{\Omega}^2), \\
I_3 &= \mathrm{tr}(\overline{S}^3), & I_4 &= \mathrm{tr}(\overline{S}\,\overline{\Omega}^2), \\
I_5 &= \mathrm{tr}(\overline{S}^2\overline{\Omega}^2), & I_6 &= \mathrm{tr}(\overline{S}^2\overline{\Omega}^2\overline{S}\,\overline{\Omega}),
\end{aligned} \tag{6.32}$$

where Id designates the identity tensor.

These tensors are independent in the sense that none can be decomposed into a linear sum of the ten others, if the coefficients are constrained to appear as polynomials of the six invariants defined above. If we relax this last constraint by considering the polynomial quotients of the invariants too, then only six of the eleven tensors are linearly independent. The tensors defined above are no longer linearly independent in two cases: when the tensor $\overline{S}$ has a double eigenvalue and when two components of the vorticity disappear when expressed in the specific reference of $\overline{S}$. The first case corresponds to an axisymmetrical shear and the second to a situation where the rotation is about a single axis aligned with one of the eigenvectors of $\overline{S}$. Assuming that neither of these conditions is verified, six of the terms of (6.31) are sufficient for representing the tensor τ, and five for representing its deviator part, which is consistent with the fact that a second-order symmetrical tensor with zero trace has only five degrees of freedom in the third dimension. We then obtain the generic polynomial form:

$$\begin{aligned}
\tau^{\mathrm{d}} &= C_1\overline{\Delta}^2|\overline{S}|\overline{S} + C_2\overline{\Delta}^2(\overline{S}^2)^{\mathrm{d}} + C_3\overline{\Delta}^2(\overline{\Omega}^2)^{\mathrm{d}} \\
&+ C_4\overline{\Delta}^2(\overline{S}\,\overline{\Omega} - \overline{\Omega}\,\overline{S}) + C_5\overline{\Delta}^2\frac{1}{|\overline{S}|}(\overline{S}^2\overline{\Omega} - \overline{S}\,\overline{\Omega}^2) \quad ,
\end{aligned} \tag{6.33}$$

where the C_i, $i = 1, 5$ are constants to be determined. This type of model is analogous in form to the non-linear statistical turbulence models [314, 315]. Numerical exper ents performed by the authors on cases of isotropic homogeneous turbulence have shown that this modeling, while yielding good results, is very costly. Also, computing the different constants raises problems because their dependence as a function of the tensor invariants involved is complex. Meneveau *et al.* [231] attempted to compute these components by statistical techniques, but achieved no significant improvement over the linear model in the prediction of the subgrid tensor eigenvectors.

We note that the first term of the expansion corresponds to subgrid viscosity models for the forward energy cascade based on large scales, which makes it possible to interpret this type of expansion as a sequence of departures from symmetry: the isotropic part of the tensor is represented by a spherical tensor, and the first term represents a first departure from symmetry but prevents the inclusion of the inequality of the normal subgrid stresses[3]. The anisotropy of the normal stresses is included by the following terms, which therefore represent a new departure from symmetry.

Kosovic's Simplified Non-Linear Model. In order to reduce the algorithmic cost of the subgrid model, Kosovic [167] proposes neglecting certain terms in the generic model presented above. After neglecting the high-order terms on the basis of an analysis of their orders of magnitude, the author proposes the following model:

$$\begin{aligned}\tau_{ij} &= -(C_s\overline{\Delta})^2\left[2(2|\overline{S}|^2)^{1/2}\overline{S}_{ij} + C_1\left(\overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij}\right)\right.\\ &+ \left. C_2\left(\overline{S}_{ik}\overline{\Omega}_{kj} - \overline{\Omega}_{ik}\overline{S}_{kj}\right)\right] \quad , \end{aligned} \tag{6.34}$$

where C_s is the constant of the subgrid viscosity model based on the large scales (see Sect. 4.3.2) and C_1 and C_2 two constants to be determined. After computation, the local equilibrium hypothesis is expressed:

$$\begin{aligned}\langle\varepsilon\rangle &= -\langle\tau_{ij}\overline{S}_{ij}\rangle \\ &= (C_s\overline{\Delta})^2\langle 2\left[(2|\overline{S}|^2)^{1/2}\overline{S}_{ij}\overline{S}_{ij} + C_1\overline{S}_{ik}\overline{S}_{kj}\overline{S}_{ji}\right]\rangle \quad . \end{aligned} \tag{6.35}$$

In the framework of the canonical case (isotropic turbulence, infinite inertial range, sharp cutoff filter), we get (see [17]):

$$\begin{aligned}\langle\overline{S}_{ij}\overline{S}_{ij}\rangle &= \frac{30}{4}\langle\left(\frac{\partial\overline{u}_1}{\partial x_1}\right)^2\rangle \\ &= \frac{3}{4}K_0\langle\varepsilon\rangle^{2/3}k_c^{4/3} \quad , \end{aligned} \tag{6.36}$$

[3] This is true for all modeling of the form $\tau = (\mathbf{V}\otimes\mathbf{V})$ in which $\mathbf{V}$ is an arbitrary vector. It is trivially verified that the tensor $(\mathbf{V}\otimes\mathbf{V})$ admits only a single non-zero eigenvalue $\lambda = (V_1^2+V_2^2+V_3^2)$, while the subgrid tensor in the most general case has three distinct eigenvalues.

$$\begin{aligned}\langle\overline{S}_{ik}\overline{S}_{kj}\overline{S}_{ji}\rangle &= \frac{105}{8}\langle\left(\frac{\partial\overline{u}_1}{\partial x_1}\right)^3\rangle \\ &= -\frac{105}{8}\mathcal{S}(k_c)\left(\frac{1}{10}K_0\right)^{3/2}\langle\varepsilon\rangle k_c^2 \quad , \end{aligned} \tag{6.37}$$

where coefficient $\mathcal{S}(k_c)$ is defined as:

$$\mathcal{S}(k_c) = -\langle\left(\frac{\partial\overline{u}_1}{\partial x_1}\right)^3\rangle/\langle\left(\frac{\partial\overline{u}_1}{\partial x_1}\right)^2\rangle^{3/2} \quad . \tag{6.38}$$

Substituting these expressions in relation (6.35) yields:

$$\langle\varepsilon\rangle = (C_s\overline{\Delta})^2\left[1 - \frac{7}{\sqrt{960}}C_1\mathcal{S}(k_c)\right]\left(\frac{3}{2}K_0\right)^{3/2}k_c^2\langle\varepsilon\rangle \quad . \tag{6.39}$$

This relation provides a way of relating the constants C_s and C_1 and thereby computing C_1 once C_s is determined by reasoning similar to that explained in the chapter on functional models. The asymptotic value of $\mathcal{S}(k_c)$ is evaluated by theory and experimental observation at between 0.4 and 0.8, as $k_c \to \infty$. The constant C_2 cannot be determined this way, since the contribution of the anti-symmetrical of the velocity gradient to the energy transfer is null[4].

On the basis of simple examples of anisotropic homogeneous turbulence, Kosovic proposes:

$$C_2 \approx C_1 \quad , \tag{6.40}$$

which completes the description of the model.

Dynamic Non-Linear Model. Kosovic's approach uses some hypotheses intrinsic to the subgrid modes, for example the existence of a theoretical the spectrum shape and the local equilibrium hypothesis. To relax these constraints, Wong [348] proposes computing the constants of the non-linear models by means of a dynamic procedure.

To do this, the author proposes a model of the form (we use the same notation here as in the description of the dynamic model with one equation for the kinetic energy, in Sect. 4.4.2):

$$\tau_{ij} = \frac{2}{3}q_{sgs}^2\delta_{ij} - 2C_1\overline{\Delta}\sqrt{q_{sgs}^2}\,\overline{S}_{ij} - C_2\overline{N}_{ij} \quad , \tag{6.41}$$

where C_1 and C_2 are constants and q_{sgs}^2 the subgrid kinetic energy, and

[4] This is because we have the relation

$$\overline{\Omega}_{ij}\overline{S}_{ij} \equiv 0 \quad ,$$

since the tensors $\overline{\Omega}$ and $\overline{S}$ are anti-symmetrical and symmetrical, respectively.

$$\overline{N}_{ij} = \overline{S}_{ik}\overline{S}_{kj} - \frac{1}{3}\overline{S}_{mn}\overline{S}_{mn}\delta_{ij} + \dot{\overline{S}}_{ij} - \frac{1}{3}\dot{\overline{S}}_{mm}\delta_{ij} \quad , \qquad (6.42)$$

where $\dot{\overline{S}}_{ij}$ is the Oldroyd[5] derivative of $\overline{S}_{ij}$:

$$\dot{\overline{S}}_{ij} = \frac{D\overline{S}_{ij}}{Dt} - \frac{\partial \overline{u}_i}{\partial x_k}\overline{S}_{kj} - \frac{\partial \overline{u}_j}{\partial x_k}\overline{S}_{ki} \quad , \qquad (6.43)$$

where D/Dt is the material derivative associated with the velocity field $\overline{\mathbf{u}}$. The isotropic part of this model is based on the kinetic energy of the subgrid modes (see Sect. 4.3.2). Usually, we introduce a test filter symbolized by a *tilde*, the cutoff length of which is denoted $\widetilde{\overline{\Delta}}$. Using the same model, the subgrid tensor corresponding to the test filter is expressed:

$$T_{ij} = \frac{2}{3}Q^2_{\rm sgs}\delta_{ij} - 2C_1\widetilde{\overline{\Delta}}\sqrt{Q^2_{\rm sgs}}\,\widetilde{\overline{S}}_{ij} - C_2\widetilde{\overline{H}}_{ij} \quad , \qquad (6.44)$$

where $Q^2_{\rm sgs}$ is the subgrid kinetic energy corresponding to the test filter, and $\widetilde{\overline{H}}_{ij}$ the tensor analogous to $\overline{N}_{ij}$, constructed from the velocity field $\widetilde{\overline{\mathbf{u}}}$. Using the two expressions (6.41) and (6.44), the Germano identity (4.126) is expressed:

$$\begin{aligned} L_{ij} &= T_{ij} - \widetilde{\tau}_{ij} \\ &\simeq \frac{2}{3}(Q^2_{\rm sgs} - \widetilde{q^2_{\rm sgs}})\delta_{ij} + 2C_1\overline{\Delta}A_{ij} + C_2\overline{\Delta}^2 B_{ij} \quad , \end{aligned} \qquad (6.45)$$

in which

$$A_{ij} = \overline{S}_{ij}\sqrt{q^2_{\rm sgs}} - \frac{\widetilde{\overline{\Delta}}}{\overline{\Delta}}\widetilde{\overline{S}}_{ij}\sqrt{Q^2_{\rm sgs}} \quad , \qquad (6.46)$$

$$B_{ij} = \widetilde{\overline{N}}_{ij} - \left(\frac{\widetilde{\overline{\Delta}}}{\overline{\Delta}}\right)^2 \widetilde{\overline{H}}_{ij} \quad . \qquad (6.47)$$

We then define the residual E_{ij}:

$$E_{ij} = L_{ij} - \frac{2}{3}(Q^2_{\rm sgs} - \widetilde{q^2_{\rm sgs}})\delta_{ij} + 2C_1\overline{\Delta}A_{ij} + C_2\overline{\Delta}^2 B_{ij} \quad . \qquad (6.48)$$

The two constants C_1 and C_2 are then computed in such a way as to minimize the scalar residual $E_{ij}E_{ij}$, *i.e.*

$$\frac{\partial E_{ij}E_{ij}}{\partial C_1} = \frac{\partial E_{ij}E_{ij}}{\partial C_2} = 0 \quad . \qquad (6.49)$$

[5] This derivative responds to the principle of objectivity, *i.e.* it is invariant if the reference system in which the motion is observed is changed

A simultaneous evaluation of these two parameters leads to:

$$2\overline{\Delta}C_1 \approx \frac{L_{mn}(A_{mn}B_{pq}B_{pq} - B_{mn}A_{pq}B_{pq})}{A_{kl}A_{kl}B_{ij}B_{ij} - (A_{ij}B_{ij})^2} \quad , \qquad (6.50)$$

$$\overline{\Delta}^2 C_2 \approx \frac{L_{mn}(B_{mn}A_{pq}A_{pq} - A_{mn}A_{pq}B_{pq})}{A_{kl}A_{kl}B_{ij}B_{ij} - (A_{ij}B_{ij})^2} \quad . \qquad (6.51)$$

The quantities $q^2_{\rm sgs}$ and $Q^2_{\rm sgs}$ are obtained by solving the corresponding evolution equations, which are described in the chapter on functional models. This completes computation of the subgrid model.

One variant that does not require the use of additional evolution equations is derived using a model based on the gradient of the resolved scales instead of one based on the subgrid kinetic energy, to describe the isotropic term. The subgrid tensor deviator is now modeled as:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2C_1\overline{\Delta}^2|\overline{S}|\overline{S}_{ij} - C_2\overline{N}_{ij} \quad . \qquad (6.52)$$

The two parameters computed by the dynamic procedure are now $\overline{\Delta}^2 C_1$ and $\overline{\Delta}^2 C_2$. The expressions obtained are identical in form to relations (6.50) and (6.51), where the tensor A_{ij} is defined as:

$$A_{ij} = |\overline{S}|\overline{S}_{ij} - |\widetilde{\overline{S}}|\widetilde{\overline{S}}_{ij}\left(\frac{\widetilde{\overline{\Delta}}}{\overline{\Delta}}\right)^2 \quad . \qquad (6.53)$$

6.1.3 Homogenization Technique: Perrier and Pironneau Models

General Description. Another category of models derived from an expansion in a small parameter is that of the models obtained by Perrier and Pironneau [263] by means of the homogenization theory. This approach, which consists in solving the evolution equations of the filtered field separately from those of the subgrid modes, is based on the assumption that the cutoff is located within the inertial range at each point. The resolved field $\overline{\mathbf{u}}$ and the subgrid field $\mathbf{u}'$ are computed on two different grids by a coupling algorithm. In all of the following, we adopt the hypothesis that $\overline{\mathbf{u}'} = 0$. The subgrid modes $\mathbf{u}'$ are then represented by a random process $\mathbf{v}^\delta$, which depends on the dissipation ε, and the viscosity ν, and which is transported by the resolved field $\overline{\mathbf{u}}$. This modeling is denoted symbolically:

$$\mathbf{u}' = \mathbf{v}^\delta\left(\varepsilon, \frac{\mathbf{x} - \overline{\mathbf{u}}t}{\delta}, \frac{t}{\delta^2}\right) \quad , \qquad (6.54)$$

in which δ^{-1} is the largest wave number in the inertial range and δ^{-2} the highest frequency cons[illegible]red. As the inertial range is assumed to extend to